Circular Viscoelastic Plates Subjected to In-Plane Loads

SAMUEL L. DELEEUW*
University of Mississippi, University, Miss.

This paper is concerned with circular viscoelastic plates subject to in-plane forces. The plates are assumed to have a small arbitrary initial curvature and the increase in curvature as a function of time is determined. The plate material is assumed to follow a linear viscoelastic stress-strain relation and quasi-static small deflection theory is utilized. The general equation for the deflection of a circular viscoelastic plate is derived from the equation of an elastic plate using the well-known correspondence principle. Special cases are: 1) a circular plate with a fixed edge; 2) a circular plate with a simply supported edge; 3) a circular segment; and 4) an annular plate. Circular symmetry is assumed in cases 1, 2 and 4 but the solution developed in case 3 could readily be applied to the other cases if symmetry did not exist. Numerical examples are given for all cases. The applied load is assumed to be uniformly distributed around the boundary of the plate. For an annular plate the load at the inner boundary may differ from the load at the outer boundary. This leads to a variation of the in-plane forces throughout the plate. The load may vary with time in any arbitrarily prescribed manner. The load is assumed to be compressive but the analysis applies equally well for tensile loads.

Introduction

THE creep buckling of plates has received considerable attention in recent years. Restricting ourselves to plates composed of linear viscoelastic materials, two papers have specifically covered this topic. Lin[1] analyzed a rectangular, simply supported, viscoelastic plate with an initial curvature and subjected to a constant in-plane force in one direction. An example was presented for a plate made of a Maxwell material. Rectangular viscoelastic plates with more general loading and boundary conditions were considered by DeLeeuw and Mase.[2] In the latter paper, plates were analyzed both with and without initial curvature. In the case of a viscoelastic plate with no initial curvature, buckling loads were defined and the method presented of determining the buckling loads applies to plates of any shape. In the case of a viscoelastic plate with initial curvature, numerical results were given for the deflection of square, simply-supported plates made of Maxwell and Kelvin materials and subjected to various in-plane loads. An extensive list of references on this topic were given in that publication.

The purpose in this paper is to analyze a circular viscoelastic plate subjected to in-plane forces. The plate is assumed to have a small arbitrary initial curvature and the increase in curvature as a function of time is determined. The plate material is assumed to follow a linear viscoelastic stress-strain relation and quasi-static small deflection theory is utilized. The general equation for the deflection of a circular viscoelastic plate is derived from the equation of an elastic plate using the well known correspondence principle. Four special cases are considered; a circular plate with a fixed edge, a circular plate with a simply supported edge, a circular segment and an annular plate. The simply supported plate is particularly interesting in that a time-dependent boundary condition occurs because of the presence of Poisson's ratio in the expression for the moment along the edge.

Equation for Circular Viscoelastic Plates

The equation in rectangular coordinates for an elastic plate with an initial curvature and subjected to in-plane forces can be found in the book by Timoshenko.[3] A transformation of this equation to polar coordinates gives

$$D\nabla^4 w_1 = N_r \frac{\partial^2(w_0 + w_1)}{\partial r^2} + N_\theta \left[\frac{1}{r}\frac{\partial(w_0 + w_1)}{\partial r} + \frac{1}{r^2}\frac{\partial^2(w_0 + w_1)}{\partial \theta^2}\right] + 2N_{r\theta}\left[\frac{1}{r}\frac{\partial^2(w_0 + w_1)}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial(w_0 + w_1)}{\partial \theta}\right] \quad (1)$$

In this equation r and θ are the space coordinates, w_0 the initial deflection and w_1 the additional deflection caused by the in-plane forces N_r, N_θ and $N_{r\theta}$. The elastic plate constant is expressed by

$$D = Eh^3/12(1 - \nu^2) = h^3G(3\kappa + G)/3(3\kappa + 4G) \quad (2)$$

where h is the plate thickness and E, ν, G and κ are the usual elastic constants. The equation for a viscoelastic plate is obtained from Eq. (1) by replacing the elastic constant D by an operator $D(p)$ in accordance with the correspondence rule stated by Biot.[4] The symbol p represents $\partial/\partial t$. In order to obtain the operator $D(p)$, consider the general stress-strain equations for a linear viscoelastic material in the operator form

$$P(p)s_{ij} = 2Q(p)e_{ij} \quad (3a)$$

$$P'(p)\sigma_{ii} = 3Q'(p)\epsilon_{ii} \quad (3b)$$

where s_{ij} and e_{ij} are the components of the deviatoric stress and strain tensors, respectively, σ_{ij} and ϵ_{ij} are the components of the stress and strain tensors, and P, Q, P' and Q' are linear operators of the form

$$P(p) = \sum_{n=0}^{q} a_n \frac{\partial^n}{\partial t^n} \qquad Q(p) = \sum_{n=0}^{s} b_n \frac{\partial^n}{\partial t^n} \quad (4a)$$

$$P'(p) = \sum_{n=0}^{q'} c_n \frac{\partial^n}{\partial t^n} \qquad Q'(p) = \sum_{n=0}^{s'} d_n \frac{\partial^n}{\partial t^n} \quad (4b)$$

The coefficients a_n, b_n, c_n and d_n are constants which represent the physical properties of the material. The operator $D(p)$ can now be obtained from Eq. (2) by replacing the elastic constants G and κ by the respective viscoelastic operators $Q(p)/P(p)$ and $Q'(p)/P'(p)$. This produces

$$D(p) = \frac{h^3Q(p)[3Q'(p)P(p) + Q(p)P'(p)]}{3P(p)[3Q'(p)P(p) + 4Q(p)P'(p)]} \quad (5)$$

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* Professor of Civil Engineering.

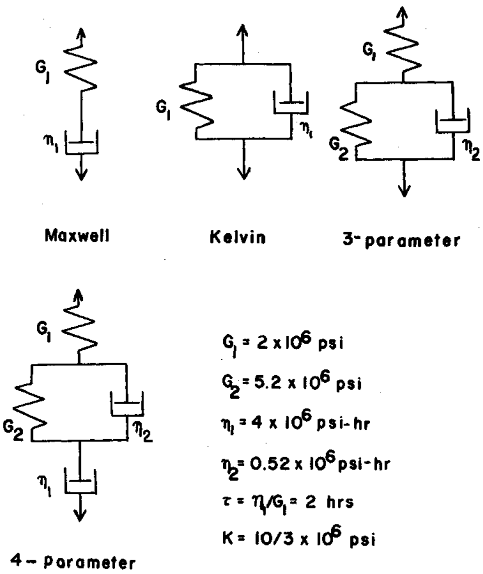

Fig. 1 Viscoelastic models used in numerical calculations.

Thus, the equation governing circular viscoelastic plates is

$$D(p)\nabla^4 w_1 = N_r \frac{\partial^2(w_0 + w_1)}{\partial r^2} + N_\theta \left[\frac{1}{r}\frac{\partial(w_0 + w_1)}{\partial r} + \frac{1}{r^2}\frac{\partial^2(w_0 + w_1)}{\partial \theta^2}\right] + 2N_{r\theta}\left[\frac{1}{r}\frac{\partial^2(w_0 + w_1)}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial(w_0 + w_1)}{\partial \theta}\right] \tag{6}$$

In the important special case where $N_r = N_\theta = -N$ and $N_{r\theta} = 0$, Eq. 6 reduces to

$$D(p)\nabla^4 w_1 = -N\nabla^2(w_0 + w_1) \tag{7}$$

Boundary Conditions

In the following sections plates with various geometries and boundary conditions are analyzed. Boundary conditions for clamped, simply supported and free edges of elastic plates were well known[3] and are the same for viscoelastic plates except when elastic constants appear in them. In such cases the elastic constants must be replaced by the corresponding viscoelastic operators which produce time dependent boundary conditions. These are not always easy to handle and the method of dealing with such problems is given in the section on the simply supported plate.

Boundary conditions for circular plates occur either along a circumferential line (r = constant) or a radial line (θ = constant). If circular symmetry exists and the plate does not have a hole at its center, a boundary condition exists there, namely; the slope of the deflected plate is zero.

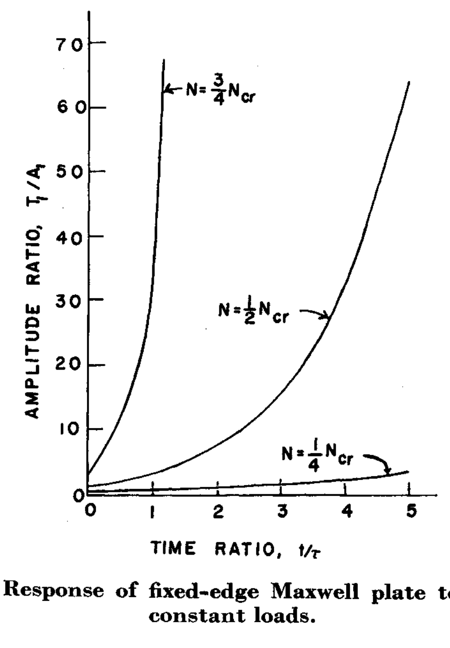

Fig. 2 Response of fixed-edge Maxwell plate to various constant loads.

Initial Conditions

Since the governing differential equation involves both space variables and time, initial conditions must be prescribed to complete the formulation of the problem. This may appear trivial as in the case of a Maxwell material where the initial deflection would be that of the elastic plate. That this is not true can be shown by an examination of Eq. (5) for a material which is hydrostatically elastic and viscoelastic in shear. This reveals that for the Maxwell model $D(p)$ is of order two, thus requiring two initial conditions. With more complex viscoelastic models the initial conditions would be even less obvious.

A general method of obtaining initial conditions for viscoelastic plate problems has been presented by the author.[2] The equations given there apply to the problems given in this paper provided they are transformed to polar coordinates.

Circular Plates with Fixed Edges

Consider a circular viscoelastic plate subjected to an inplane force distributed uniformly around its boundary which is fixed. The buckling load for an elastic circular plate[3] is obtained by assuming the deflected shape of the plate in the form of a Bessel function of order zero. Therefore, it follows that the deflection of a circular plate with arbitrary initial curvature can be expressed as a series of Bessel functions, provided the initial curvature can be expressed in a similar

Table 1 Results for the amplitude ratios $T_n(t)/A_n$ for a load $(\frac{1}{2})$ N_{cr}

Time ratio t/τ	Amplitude ratio T_1/A_1	T_2/A_2	T_3/A_3	T_4/A_4	T_5/A_5
		Maxwell			
0	1.000	0.175	0.076	0.043	0.028
1	3.058	0.330	0.136	0.075	0.048
2	7.133	0.500	0.197	0.108	0.068
3	15.259	0.689	0.260	0.140	0.088
4	31.481	0.901	0.326	0.174	0.109
5	63.882	1.140	0.396	0.208	0.130
		Kelvin			
0	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.5369	0.1239	0.0560	0.0320	0.0208
2	0.7441	0.1544	0.0685	0.0389	0.0252
3	0.8583	0.1667	0.0733	0.0415	0.0268
4	0.9215	0.1718	0.0752	0.0425	0.0274
5	0.9566	0.1739	0.0759	0.0429	0.0277
		3-Parameter			
0	1.000	0.175	0.076	0.043	0.028
0.1	1.578	0.226	0.096	0.054	0.035
0.2	1.710	0.233	0.099	0.055	0.036
0.3	1.740	0.234	0.099	0.056	0.036
0.4	1.747	0.234	0.099	0.056	0.036
0.5	1.749	0.234	0.099	0.056	0.036
		4-Parameter			
0	1.000	0.175	0.076	0.043	0.028
1	5.801	0.401	0.160	0.088	0.056
2	16.229	0.588	0.223	0.121	0.076
3	42.583	0.798	0.289	0.154	0.097
4	109.213	1.036	0.359	0.188	0.118
5	277.685	1.304	0.432	0.224	0.139
		Elastic			
	1.000	0.175	0.076	0.043	0.028

series. If the plate material is viscoelastic, the series will have time dependent coefficients. Assuming circular symmetry, the initial deflection can be expressed as

$$w_0(r) = \Sigma_n A_n J_0(\lambda_n r) + A_0 \quad (8)$$

where J_0 represents a Bessel function of order zero. Also, the solution of Eq. (7) can be expressed in the form

$$w_1(r,t) = \Sigma_n T_n(t) J_0(\lambda_n r) + F(t) \quad (9)$$

where the $T_n(t)$'s are time dependent coefficients and $F(t)$ is an arbitrary function of time. Substitution of Eqs. (8) and (9) into Eq. (7) and following the usual Fourier-Bessel analysis[5] produces

$$[\lambda_n^2 D(p) - N] T_n(t) = N A_n \quad (10)$$

which represents a linear, ordinary differential equation for each of the $T_n(t)$'s.

The boundary conditions for a fixed-edge circular plate are

$$w_1]_{r=a} = 0 \quad (11a)$$

$$\partial w_1/\partial r]_{r=a} = 0 \quad (11b)$$

$$\partial w_1/\partial r]_{r=0} = 0 \quad (11c)$$

where $r = a$ represents the boundary of the plate. Substitution of Eq. (9) into Eq. (11a) determines $F(t)$ as

$$F(t) = -\Sigma_n T_n(t) J_0(\lambda_n a) \quad (12)$$

The second boundary condition, Eq. (11b), is satisfied if

$$\{dJ_0(\lambda_n r)/dr\}]_{r=a} = 0 \quad (13)$$

which is equivalent to[5]

$$J_1(\lambda_n a) = 0 \quad (14)$$

where J_1 represents a Bessel function of order one. Similarly, the third boundary condition, Eq. (11c), which is due to symmetry is satisfied because $J_1(0) = 0$. Substitution of Eq. (12) into Eq. (9) gives the solution in the form

$$w_1(r,t) = \Sigma_n T_n(t)[J_0(\lambda_n r) - J_0(\lambda_n a)] \quad (15)$$

where the λ_n's are defined by Eq. (14) and the $T_n(t)$'s by Eq. (10).

As a numerical example, Eq. (10) was solved for the case when N is constant. The equation has constant coefficients under this loading condition and thus, the solution of Eq. (10) can be expressed in exponential form. The plate material was assumed viscoelastic in shear and hydrostatically elastic. Results were obtained for the viscoelastic models shown in Fig. 1. The expression N_{cr} represents the buckling load corresponding to a circular, fixed edge, elastic plate with shear modulus G_1. Results for the amplitude ratios $T_n(t)/A_n$ are presented in Table 1 for a load equal to $(\frac{1}{2})N_{cr}$. It is noticed that the first term is quite dominant as time increases. The deflection of an elastic plate with shear constant G_1 is included for comparison purposes and also to check the results of the viscoelastic plates. The initial response of each viscoelastic plate except the Kelvin plate should equal the response of the elastic plate. The deflection of the Kelvin plate should approach the elastic plate deflection as time approaches infinity. The results in Table 1 verify these statements. Figures 2 and 3 present the ratios $T_1(t)/A_1$ for the Maxwell and Kelvin models for various values of the load.

Table 2 Adams method and power series results for amplitude ratios

	Amplitude ratios, T_1/A_1					
Time ratio, t/τ	0	1	2	3	4	5
Adams method	0.00000	0.02533	0.13080	0.40535	1.13499	3.73026
Power series	0.00000	0.02533	0.13079	0.40532	1.13487	3.72934

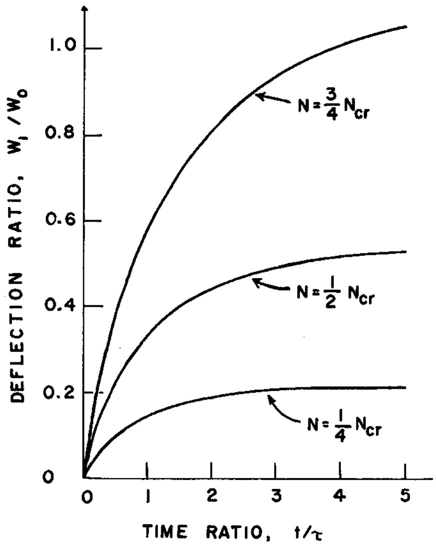

Fig. 3 Response of fixed-edge Kelvin plate to various constant loads.

As a second numerical example, Eq. (10) was solved for the case where N varies with time. Under this condition Eq. (10) has variable coefficients. In particular the load was assumed in the form $N = ct^2$ where the constant c was chosen such that $N = (\frac{1}{2})N_{cr}$. when $t = 10$ hours. Adam's numerical technique of solving differential equations was employed using an IBM SHARE program written by R. D. Glaus of the Aeroject General Corporation. As a check, a power series solution was also programmed for a Maxwell material and comparison of the two methods is given in Table 2. Figures 4 and 5 give the response of Maxwell and Kelvin plates subjected to the above load. For comparison purposes the response of an elastic plate with shear constant G_1 is also shown.

Simply Supported Circular Plates

The boundary conditions for a simply-supported, circular, viscoelastic plate are

$$w_1]_{r=a} = 0 \quad (16a)$$

$$M_r]_{r=a} = D(p)[\partial^2 w_1/\partial r^2 + (\nu(p)/r)(\partial w_1/\partial r)]_{r=a} = 0 \quad (16b)$$

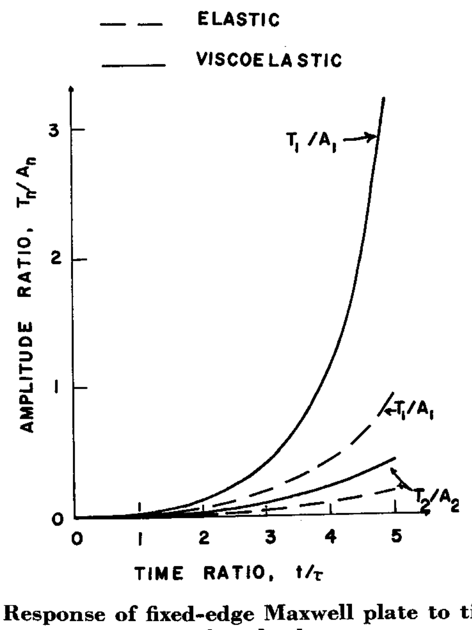

Fig. 4 Response of fixed-edge Maxwell plate to time-dependent load.

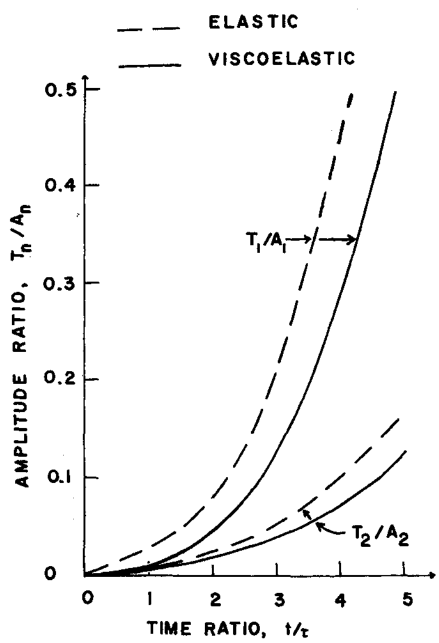

Fig. 5 Response of fixed-edge Kelvin plate to time-dependent load.

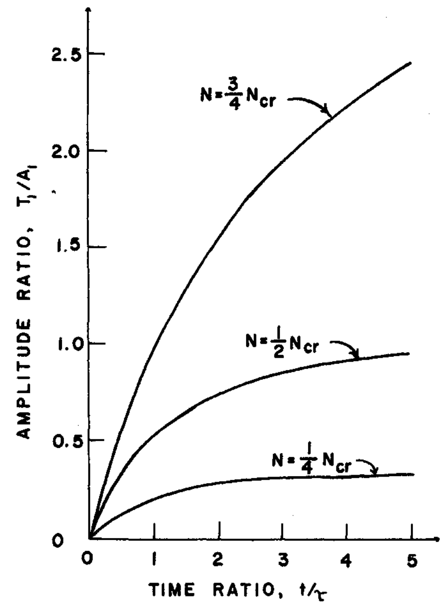

Fig. 7 Response of simply supported Kelvin plate to various constant loads.

$$\partial w_1/\partial r]_{r=0} = 0 \qquad (16c)$$

The plate moment in the radial direction M_r was obtained through the use of the correspondence rule.[4] It can be verified[2] that the operator corresponding to Poisson's ratio in accordance with Eqs. (3) can be expressed in the form

$$\nu(p) = \frac{3Q'(p)P(p) - 2Q(p)P'(p)}{2[3Q'(p)P(p) + Q(p)P'(p)]} \qquad (17)$$

Thus, the boundary condition for M_r is time dependent in that it is expressed as a differential equation in time. A solution of the form of Eq. (9) will not be suitable with such a boundary condition. A method of handling time dependent boundary conditions in vibration problems[6] can be utilized here. Let the initial deflection and solution of Eq. (10) be assumed in the forms

$$w_0(r) = \Sigma_m A_m J_0(\lambda_m r) \qquad (18a)$$

$$w_1(r,t) = \Sigma_n T_n(t) J_0(\lambda_n r) + F_1(t) + r^2 F_2(t) \qquad (18b)$$

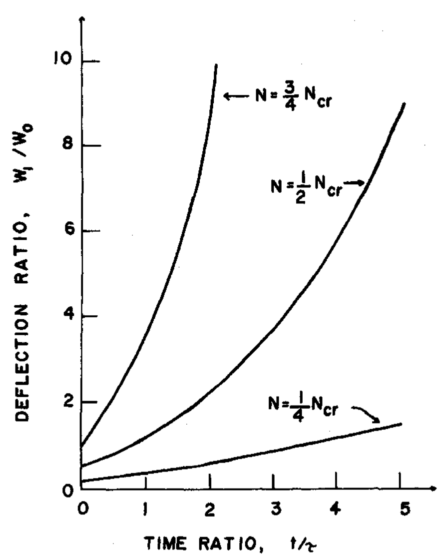

Fig. 6 Response of simply supported Maxwell plate to various constant loads.

where $F_1(t)$ and $F_2(t)$ are arbitrary functions of time. It is seen that the boundary condition at the center of the plate is satisfied, while satisfaction of the condition that the deflection at $r = a$ be zero can be accomplished by setting

$$J_0(\lambda_n a) = 0 \qquad (19)$$

and

$$F_1(t) = -a^2 F_2(t) \qquad (20)$$

Substitution of Eq. (20) into Eq. (18b) gives the solution in the form

$$w_1(r,t) = \Sigma_n T_n(t) J_0(\lambda_n r) + (r^2 - a^2) F_2(t) \qquad (21)$$

When Eq. (21) is substituted into the boundary condition on M_r, Eq. (16b), the following differential equation is obtained.

$$D(p)\{\Sigma_n \lambda_n J_1(\lambda_n a)[1 - \nu(p)]T_n(t) + 2a[1 + \nu(p)]F_2(t)\} = 0 \qquad (22)$$

Expanding $(r^2 - a^2)$ into a Fourier-Bessel series, substitution of this series and Eqs. (18a) and (21) into Eq. (7) and following the usual Fourier-Bessel analysis produces

$$[\lambda_n^2 D(p) - N]\{T_n(t) - [8F_2(t)/a\lambda_n^3 J_1(\lambda_n a)]\} = NA_n \qquad (23)$$

If M terms are taken in the series of Eq. (18b), Eqs. (22) and (23) provide $M + 1$ linear, ordinary, simultaneous, differential equations for the $T_n(t)$'s and $F_2(t)$.

As a numerical example N was again assumed to be constant with time. Then, Eqs. (22) and (23) represent simultaneous differential equations with constant coefficients. In the expansion of Eq. (18a), all of the A_m's were assumed zero except A_1 and solutions were obtained using one, two and three terms in the expansion of Eq. (21). With three terms, convergence was within 1.4%. The deflection ratio w_1/w_0 at the center of the plate for the Maxwell and Kelvin models is represented in Figs. 6 and 7 for various values of the load. The expression N_{cr}. here represents the buckling load of a simply supported, circular, elastic plate with shear modulus G_1.

Circular Segment

Consider a plate in the shape of a circular segment acted upon by an in-plane load evenly distributed around its

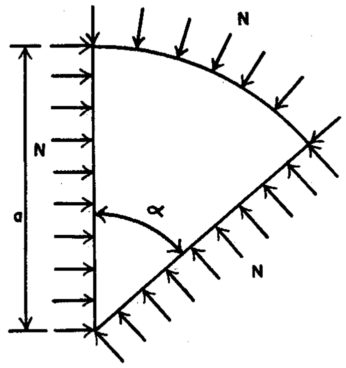

Fig. 8 Circular segment subjected to in-plane load.

boundary as illustrated in Fig. 8. It is assumed that the edges $\theta = 0$ and $\theta = \alpha$ are simply supported and the edge $r = a$ fixed. The buckling loads for an elastic segment were determined by Galerkin.[7] The boundary conditions are

$$w_1 \Big]_{\substack{\theta=0\\ \theta=\alpha}} = 0 \tag{24a}$$

$$M_\theta \Big]_{\substack{\theta=0\\ \theta=\alpha}} = D(p)\left[\frac{1}{r}\frac{\partial w_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_1}{\partial \theta^2} + \nu(p)\frac{\partial^2 w_1}{\partial r^2}\right]_{\substack{\theta=0\\ \theta=\alpha}} = 0 \tag{24b}$$

$$w_1]_{r=a} = 0 \tag{24c}$$

$$\partial w_1/\partial r]_{r=a} = 0 \tag{24d}$$

where M_θ is the plate moment in the θ direction. The initial deflection and the additional deflection for the viscoelastic case can be expressed as

$$w_0(r,\theta) = \Sigma_n\Sigma_m A_{mn} J_{n\pi/\alpha}(\lambda_{mn}r)\sin(n\pi\theta/\alpha) \tag{25a}$$

$$w_1(r,\theta,t) = \Sigma_n\{[\Sigma_m T_{mn}(t)J_{n\pi/\alpha}(\lambda_{mn}r)] + F_n(t)r^{n\pi/\alpha}\}\sin(n\pi\theta/\alpha) \tag{25b}$$

where $J_{n\pi/\alpha}$ is a Bessel function and $F_n(t)$ is an arbitrary function of time. The boundary conditions at $\theta = 0$ and $\theta = \alpha$ are identically satisfied, while the boundary conditions at $r = a$ when solved simultaneously yield

$$(2n\pi/\alpha)J_{n\pi/\alpha}(\lambda_{mn}a) - \lambda_{mn}aJ_{(n\pi-\alpha)/\alpha}(\lambda_{mn}a) = 0 \tag{26a}$$

$$F_n(t) = (-1/a^{n\pi/\alpha})\,\Sigma_m T_{mn}(t)J_{n\pi/\alpha}(\lambda_{mn}a) \tag{26b}$$

Equation (26a) is the identical frequency equation for λ_{mn} obtained by Galerkin.[7] Equation (26b) can be substituted into Eq. (25b) which then gives the solution as

$$w_1(r,\theta,t) = \Sigma_n\Sigma_m T_{mn}(t)[J_{n\pi/\alpha}(\lambda_{mn}r) - (r/a)^{n\pi/\alpha}J_{n\pi/\alpha}(\lambda_{mn}a)]\sin(n\pi\theta/\alpha) \tag{27}$$

Substitution of Eqs. (25a) and (27) into Eq. (7) determines the following linear, ordinary, differential equations for the $T_{mn}(t)$'s.

$$[\lambda_{mn}{}^2D(p) - N]T_{mn}(t) = NA_{mn} \tag{28}$$

It is noted that this has the same form as Eq. (10) which pertains to a complete circular plate with a fixed edge, if λ_{mn} and A_{mn} are replaced by λ_n and A_n respectively. Ratios of T_{11}/A_{11} for Maxwell and Kelvin plates are shown in Figs. 9 and 10 for $\alpha = \pi/2$.

For segments with different boundary conditions a method similar to that discussed in the previous section can be used. The method of this section is also applicable to complete circular plates where the initial curvature is not symmetrical. In such a case, the initial deflection and the solution can be expressed in the form of Eqs. (25).

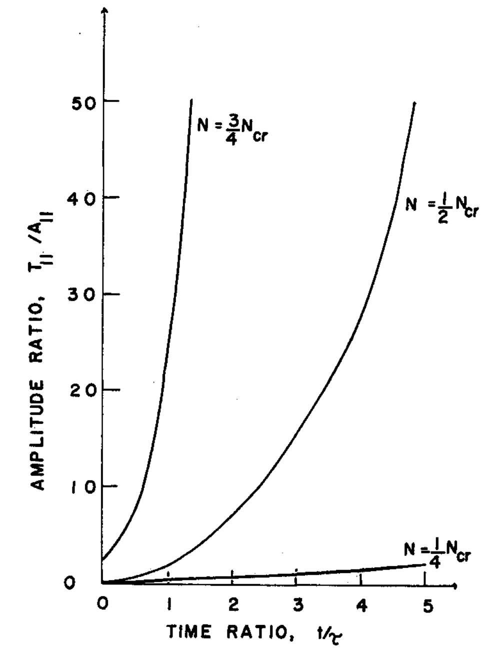

Fig. 9 Response of Maxwell segment to various constant loads.

Annular Plates

The buckling loads for an elastic annular plate subjected to in-plane forces on the inner ($r = b$) and outer ($r = a$) edges were determined by Meissner.[8] Here the basic difficulty is that the in-plane forces N_r and N_θ vary throughout the plate. Since the plate is basically flat, Lame's solution for a thick walled elastic cylinder can be utilized to represent the variation of N_r and N_θ throughout the plate. Further, the expressions for N_r and N_θ which have the general form

$$N_r = A - B/r^2 \tag{29a}$$

$$N_\theta = A + B/r^2 \tag{29b}$$

are independent of the elastic constants and thus, are also valid for a viscoelastic plate. Expressions for A and B are well known and can be found in a text such as Timoshenko.[9] A procedure similar to Meissner's can be followed whereby first, the expressions for N_r and N_θ are substituted into Eq.

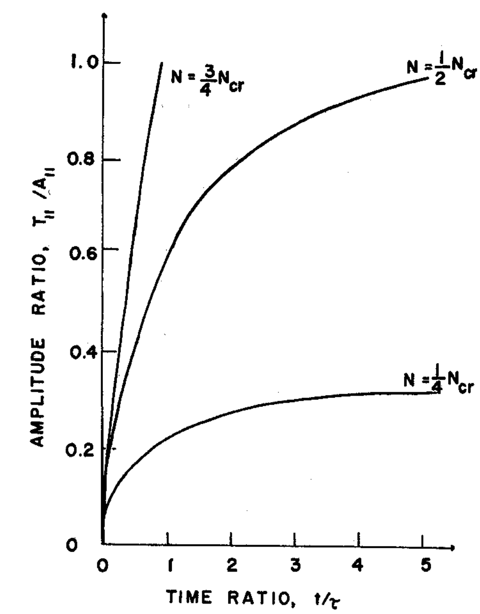

Fig. 10 Response of Kelvin segment to various constant loads.

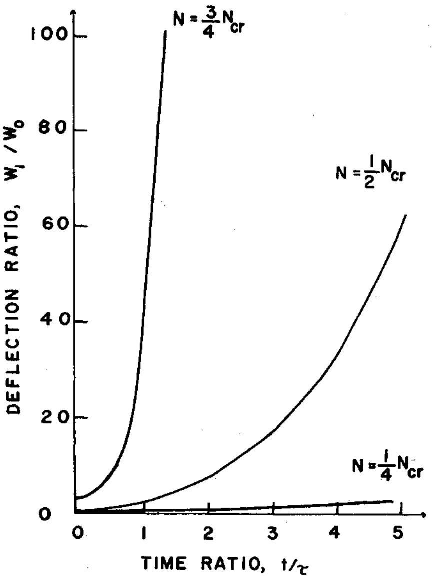

Fig. 11 Response of Maxwell annular plate to various constant loads.

(6); second, the equation is integrated once; and third, the expressions $\partial w_0/\partial r$ and $\partial w_1/\partial r$ are replaced by ϕ_0 and ϕ_1, respectively. This procedure gives

$$D(p)\left[\frac{\partial^2\phi_1}{\partial r^2} + \frac{1}{r}\frac{\partial\phi_1}{\partial r} - \frac{1}{r^2}\phi_1\right] - \left[A - \frac{B}{r^2}\right]\phi_1 = \left[A - \frac{B}{r^2}\right]\phi_0 + \frac{F_1(t)}{r} \quad (30)$$

Physically, $\phi = \phi_1 + \phi_0$ represents the slope of the deflection curve in the radial direction. Again circular symmetry is assumed. Galerkin's[10] method is now employed. The initial curvature and solution can be assumed in the forms

$$\phi_0 = \Sigma_n A_n Z_1(\lambda_n r) \quad (31a)$$

$$\phi_1 = \Sigma_n T_n(t) Z_1(\lambda_n r) \quad (31b)$$

where

$$Z_1(\lambda_n r) = J_1(\lambda_n r) + B_n Y_1(\lambda_n r) \quad (32)$$

Y_1 is a Bessel function of the second kind of order one. The last term of Eq. (30) can be expanded to

$$F_1(t)/r = F_1(t)\Sigma_n C_n Z_1(\lambda_n r) \quad (33)$$

Substitution of Eqs. (31) and (33) into Eq. (30), applying Galerkin's method and making use of the orthoganality of the Bessel functions when possible gives

$$D_m[\lambda_m{}^2 D(p) + A]T_m(t) - B\Sigma_n E_{mn} T_n(t) = B\Sigma_n A_n E_{mn} - D_m[AA_m + C_m F_1(t)] \quad (34)$$

where

$$D_m = \int_b^a r[Z_1(\lambda_m r)]^2 dr \quad (35a)$$

$$C_m = \frac{1}{D_m}\int_b^a Z_1(\lambda_m r)dr = \frac{1}{\lambda_m D_m}[Z_0(\lambda_m b) - Z_0(\lambda_m a)] \quad (35b)$$

$$E_{mn} = \int_b^a (1/r)Z_1(\lambda_m r)Z_1(\lambda_n r)dr \quad (35c)$$

and

$$Z_0(\lambda_n r) = J_0(\lambda_n r) + B_n Y_0(\lambda_n r) \quad (36)$$

Y_0 represents a Bessel function of the second kind of order zero.

Consider now a plate which is clamped at both the inner ($r = b$) and outer ($r = a$) boundaries. The boundary conditions are then

$$\phi_1\Big]_{r=b}^{r=a} = 0 \quad (37)$$

The substitution of Eq. (31b) into Eq. (37) will lead to the definitions of λ_n and B_n in Eq. (32).

To obtain the deflection, Eq. (31b) can be integrated to

$$w_1(r,t) = \int \Sigma_n T_n(t) Z_1(\lambda_n r)dr + F_2(t) = -\Sigma_n(1/\lambda_n)T_n(t)Z_0(\lambda_n r) + F_2(t) \quad (38)$$

The deflection at $r = b$ must be zero which leads to

$$F_2(t) = \Sigma_n(1/\lambda_n)T_n(t)Z_0(\lambda_n b) \quad (39)$$

Also, the deflection at $r = a$ must be zero. Thus,

$$\Sigma_n(1/\lambda_n)T_n(t)[Z_0(\lambda_n b) - Z_0(\lambda_n a)] = 0 \quad (40)$$

If M terms are taken in the expansion of Eq. (31b), Eq. (34) provides M simultaneous, linear, ordinary differential equations for the $T_n(t)$'s and $F_1(t)$. Equation (40) gives the other required equation. Using Eq. (39), the deflection can now be written as

$$w_1(r,t) = \Sigma_n(1/\lambda_n)T_n(t)[Z_0(\lambda_n b) - Z_0(\lambda_n r)] \quad (41)$$

The results of a two-term series are shown in Fig. 11. Here the deflection amplitude w_1/w_0 is given as a function of time for a Maxwell plate.

Conclusions

Circular viscoelastic plates with an initial curvature and subjected to in-plane forces were analyzed. Small deflection, quasi-static theory was used. In addition to complete circular plates, circular segments and annular plates were also considered. The governing differential equations were derived using a correspondence rule and solutions for the plate deflections were determined in the form of a series of Bessel functions with time dependent coefficients. Numerical examples were presented for several cases assuming the applied force to be constant and one case where the load varied with time. In these examples, the plate material was assumed to be hydrostatically elastic and linearly viscoelastic in shear.

Generally, the first term of the Bessel function series was quite dominant provided the first term of the series representing the initial deflection dominated. In the case of a fixed edge plate subjected to constant load the maximum difference between the ratios $T_1(t)/A_1$ and $T_2(t)/A_2$ was 23%. Assuming a difference between A_1 and A_2 of the same magnitude, a one-term approximation when compared with a two-term approximation would be accurate to within 5.3%. Following the same type of reasoning a one-term approximation for the case of a fixed-edge plate with a variable load gives accuracy to within 9%. Including more terms in the series provides no computational difficulties since each $T_n(t)$ is independent of the other $T_n(t)$'s. Results for approximations up to five terms are presented in Table 1.

Similar results are obtained for the other numerical examples. In the cases of the simply supported and annular plates, including more terms in the series approximation presents increased difficulty since the equations defining the $T_n(t)$'s are simultaneous. Generally, the results in this paper indicate that the effect of the higher harmonics decreases with time.

In general, it has been demonstrated in this paper that the problem of a circular, linear viscoelastic plate subjected to in-plane forces can be reduced to solving a set of simultaneous, linear, ordinary differential equations for the time dependent coefficients of a series of Bessel functions. In some cases the equations are not simultaneous. If the applied loads are constant in time, the equations have constant coefficients and the solutions can be put into exponential form. Otherwise a power series solution or a numerical procedure is necessary.

References

[1] Lin, T. H., "Creep Deflection of Viscoelastic Plate Under Uniform Edge Compression," *Journal of the Aeronautical Sciences*, Vol. 23, No. 9, Sept. 1956, pp. 883–887.

[2] DeLeeuw, S. L., and Mase, G. E., "Behavior of Viscoelastic Plates Under the Action of In-Plane Forces," *Proceedings of the Fourth U.S. National Congress of Applied Mechanics*, June, 1962, pp. 999–1005.

[3] Timoshenko, S., *Theory of Elastic Stability*, McGraw-Hill, New York, 1936.

[4] Biot, M. A., "Variational and Lagrangian Methods in Viscoelasticity," *International Union of Theoretical and Applied Mechanics Colloquium on Deformation and Flow of Solids*, Sept. 1955, pp. 251–263.

[5] Hildebrand, F. B., *Advanced Calculus for Engineers*, Prentice-Hall, New York, 1948.

[6] Mindlin, R. D., and Goodman, L. E., "Beam Vibrations with Time-Dependent Boundary Conditions," *Transactions of the American Society of Mechanical Engineers, Journal of Applied Mechanics*, Vol. 72, 1950, pp. 377–380.

[7] Galerkin, B., "Sur la Stabilité d'une Plaque Uniformément Comprimée Parallèlment a su Surface, Limitée par Deux Arcs de Cercles Concentriques et par Deux Rayons," *Comptes Rendus de L'Academie des Sciences*, Vol. 179, 1924, pp. 1392–1394.

[8] Meissner, E., "Ueber das Knicken Kreisringformiger Scheiben," *Schweizerische Bauzeitung*, Vol. 101, Feb. 1933, pp. 87–89.

[9] Timoshenko, S., *Strength of Materials*, Pt. II, Van Nostrand, New York, 1941.

[10] Galerkin, B., "Series Solutions of Some Problems of Elastic Equilibrium of Rods and Plates," *Vestnik Inzhenerov*, Vol. 1, 1915, pp. 879–908.

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Stress Waves in Layered Thermoelastic Media Generated by Impulsive Energy Deposition

H. E. GASCOIGNE*
University of Utah, Salt Lake City, Utah

AND

I. K. McIVOR†
University of Michigan, Ann Arbor, Mich.

When high-intensity electromagnetic energy impinges on an elastic media, energy is deposited internally by photoelectric absorption and scattering. A spatially nonuniform heating is produced with deformation waves being generated in the media. The transient stresses resulting from such an energy deposition in an initially undisturbed layered media is investigated. The internal energy deposition is considered to be equivalent to an internal heat generation varying exponentially with the distance from the exposed surface. Energy is deposited internally according to a different law in each layer. The energy deposition is assumed to occur in a time negligible in comparison to a characteristic mechanical response time. Only spatial variations normal to the plane of the layers are considered. The equations of linear, uncoupled thermoelasticity, assuming constant material properties, are applied to a two-layer media. A boundary value problem is formulated and solved exactly neglecting heat conduction. Numerical examples are presented for several cases showing the effect of specific energy deposition functions, material thickness ratio, and material parameters. A complete description of interior stress reflection is presented. The influence of energy deposition and other material parameters on the interfacial bond stress is shown. Relatively small changes in layer parameters may have a marked effect on the maximum stress, depending on the manner in which the generated stress waves combine.

Introduction

MOST solids exhibit some opacity to electromagnetic radiation with energy being deposited in the media as a result of photoelectric absorption and scattering. Energy is propagated into the media at velocities comparable to the speed of light with, in some cases, a time pulse width such that deposition is complete in times on the order of 10^{-8} sec. This input time is substantially less than the time for an elastic dilatation wave to travel through the medium a characteristic distance. For metallic structures, using a characteristic length of 1 cm, this typical mechanical response time is on the order of 10^{-6} sec. The deposition of energy as a function of depth penetrated depends on the energy spectra of the radiation. The deposition (or absorption) law is an exponential decreasing function for monochromatic radiation.

Because of the extremely short-time duration of deposition, elastic deformation waves are generated. White[1] studied the effects of absorption of radiation created by high-power, pulsed light sources. Zaker[2] investigated elastic stress wave generation in an elastic half-space and in a homogeneous elastic slab caused by absorption of energy at microwave frequencies. Morland[3] studied the effect of electromagnetic absorption in an elastic half-space; Hegemier and Morland[4] investigated the effect in a viscoelastic half-space. No attempt has been made in these investigations to include the effect of nonhomogeneous media.

Many modern aerospace structures are composed of plate and shell-like sections formed as laminates with two or more layers of different materials. They are joined either by molecular bonding or by a bond with finite thickness and elasticity. Adhesion failure due to transient loading produc-

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* Assistant Professor, Department of Mechanical Engineering.

† Professor, Department of Engineering Mechanics.